On Solving an Amusing Puzzle

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Abstract:

The study of Diophantine equations has been of great interest to mathematicians of long time. In this paper, we will introduce an amusing puzzle, and solve it using the idea of continued fraction method. It turns out that the puzzle gives rise to a quadratic Diophantine equation. We will be solving this equation using a nice continued fraction. Finally, we have obtained closed expressions forming infinitely many solutions to the given puzzle.

Keywords: Quadratic Diophantine Equation, Continued Fraction, Convergent, Recurrence Relations, Characteristic Equation

1. Introduction

Solving equations whose solutions are integers are traditionally known as Diophantine Equations. In this paper, we will introduce an amusing puzzle, which reduces to a quadratic Diophantine equation. To solve this, we had used a nice continued fraction which generates required solutions to the given puzzle. Finally, we had framed recurrence relations regarding the solutions obtained and had derived nice closed formulas for expressing the infinitely many solutions pertaining to the puzzle.

2. Description of the Puzzle

The primary aim of this paper is to solve the following puzzle:

Find all possible positive integers such that either one added to its thrice is a perfect square or one added to its five times is also a perfect square (1). If k is one of the positive integer satisfying the conditions of the puzzle as posed in (1), then we get $3k + 1 = x^2$ (2) and $5k + 1$

= y^2 (3) for some integers x and y. Our objective is to determine all possible k such that equations (2) and (3) are simultaneously true.

3. Methods of Solution

From (2) and (3), we notice that $5x^2 - 3y^2 = 2$ (4). Thus in order to solve the given puzzle, it is enough to solve equation (4). Equation (4) is a special of equation known as quadratic Diophantine equations. In the next section, we would be presenting the methods to solve (4).

4. Solving the Puzzle

In order to solve equation (4) namely, $5x^2 - 3y^2 = 2$, we first note that $\left(\sqrt{5}-\sqrt{3}\right)\times\left(\sqrt{5}+\sqrt{3}\right)=2$ Using this simple observation, we consider the following computations

$$\sqrt{5}-\sqrt{3}=\frac{2}{\sqrt{5}+\sqrt{3}}=\frac{2}{2\sqrt{5}-(\sqrt{5}-\sqrt{3})}=\frac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-(\sqrt{5}-\sqrt{3})}}$$

$$=\frac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-(\sqrt{5}-\sqrt{3})}}}=\cdots=\frac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cdots}}}}}$$

Hence, we obtain

$$\sqrt{3}=\sqrt{5}-\frac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cdots}}}}} \qquad (5)$$

We now compute the successive convergents from the continued fraction obtained in (5).

$$\frac{\sqrt{5}}{1},\sqrt{5}-\frac{2}{2\sqrt{5}}=\frac{4}{\sqrt{5}},\sqrt{5}-\frac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}}}=\frac{7\sqrt{5}}{9},\sqrt{5}-\frac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}}}}=\frac{31}{8\sqrt{5}},$$

$$\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}}}}}=\frac{55\sqrt{5}}{71},\ \sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}}}}}}=\frac{244}{63\sqrt{5}},$$

$$\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}-\cfrac{2}{2\sqrt{5}}}}}}}=\frac{433\sqrt{5}}{559},\ \ldots$$

Considering the first, third, fifth, seventh, ninth, eleventh, in general odd order convergents from above computations, then they can be written in the form $\frac{x\sqrt{5}}{y}$ were (x, y) are solutions to $5x^2-3y^2=2$.

In doing so, we see that the solutions to $5x^2-3y^2=2$ are given by $(x,y)=(1,1);(7,9);(55,71);(433,559);(3409,4401);(26839,34649);\ldots$ (6)

Now from (2) and (3), the solutions to the original puzzle are given by $k=\frac{x^2-1}{3}=\frac{y^2-1}{5}$ were (x, y) are given by (6). Thus the numbers satisfying the given puzzle as described in (1) are

16, 1008, 62496, 3873760, 240110640, . . . (7)

5. Closed Form of the Solutions

In this section, we will determine the closed form of solutions as listed in (7) to the puzzle (1).

For doing this, first we observe that the solutions to $5x^2-3y^2=2$ are given by the recurrence relations $x_{n+2}=8x_{n+1}-x_n,\ y_{n+2}=8y_{n+1}-y_n$ (8) were $(x_0,y_0)=(1,1);(x_1,y_1)=(7,9)$.

The characteristic equations corresponding to the recurrence relations in (8) is given by $m^2-8m+1=0$ (9).

The two solutions of the characteristic equation are given by $\alpha=4+\sqrt{15},\beta=4-\sqrt{15}$ (10). From (10), we observe that $\alpha+\beta=8,\alpha-\beta=2\sqrt{15},\alpha\beta=1$ (11)

Thus for $n \geq 0$, the solutions of (8) are given by $x_n = c_1\alpha^n + c_2\beta^n$, $y_n = k_1\alpha^n + k_2\beta^n$ (12)

Now using the initial conditions $(x_0, y_0) = (1,1); (x_1, y_1) = (7,9)$ and (11), we find that the general solutions to the equation $5x^2 - 3y^2 = 2$ can be expressed in the form

$$x_n = \left(\frac{\alpha^n + \beta^n}{2}\right) + \frac{3}{2\sqrt{15}}\left(\alpha^n - \beta^n\right) \quad (13)$$

$$y_n = \left(\frac{\alpha^n + \beta^n}{2}\right) + \frac{5}{2\sqrt{15}}\left(\alpha^n - \beta^n\right) \quad (14)$$

Thus for $n \geq 1$, the required numbers forming the solutions to the puzzle described are positive integers *k*, were $k = \frac{x_n^{\,2} - 1}{3} = \frac{y_n^{\,2} - 1}{3}$ (15)

We notice that x_n, y_n in (15) were given by the expressions (13) and (14) respectively.

6. Conclusion

By introducing an interesting and amusing puzzle as described in (1) of this paper, we had transformed in to an equivalent form representing quadratic Diophantine equation. To solve this transformed equation, we had described a continued fraction expression as in (5) in a novel way. Taking the alternate convergents of this continued fraction, we can generate infinitely many solutions of the quadratic Diophantine equation $5x^2 - 3y^2 = 2$.

Using these solutions, we can immediately obtain solutions to the given puzzle as described through equation (8). Finally, to obtain the closed form expressions for the solutions to the quadratic Diophantine equation, we had constructed two identical recurrence relations as in (8), whose solutions are provided in (13) and (14) respectively. Finally, using these two expressions, the required positive integers forming solutions to the puzzle are precisely the numbers given by equation (15). Thus, we have solved the given puzzle completely using continued fraction and recurrence relations in a way as elegant as possible.

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