Generalized Fixed Point Theorems of Ciric Type in Fuzzy Metric Spaces

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Abstract:

The present paper is the investigation of possibilities for improvements and generalizations contractive condition of Ciri´c in the fuzzy metric spaces. Various versions of fuzzy ´ contractive conditions are studied in two directions. Establish fixed point theorems for quasi-contractive mappings and for $\mathcal{H}-$ contractive mappings more general contractive conditions in fuzzy metric spaces are achieved and secondly, quasi-contractive type of mappings are investigated in order to obtain fixed point results with a wider class of t-norms.

Keywords: fixed point, common fixed point , fuzzy quasi – contractive mapping fuzzy metric space; t-norm; quasi-contractive mapping

1. Introduction

Famous Banach and Edelstein results have fundamental role in many fixed point theorems It is well known that the fuzzy metric spaces are a generalization of the metric spaces, based on the theory of fuzzy sets [30]. Kramosil and Michalek [22] introduced a fuzzy metric spaces performing the probabilistic metric spaces approach to the fuzzy settings. Further on, George and Veeramani [13], [14] obtained a Hausdorff topology for specific fuzzy metric spaces with important applications in quantum physics [11], [12]. Accordingly, many authors translated the various contraction mappings from metric to fuzzy metric spaces,

The Banach contraction principle [1] is usually taken as a starting point for many studies in the fixed point theory. The principle is observed in various types of metric spaces, as well as different generalizations of it.

One of the most cited generalizations of the Banach contraction principle in probabilistic metric spaces is by Ciri´c [´ 11]. More information about the fuzzy and probabilistic metric spaces, as well as fixed point theory in these spaces, can be found in [12–18].

First, we list basic definitions and propositions about t-norms and fuzzy metric spaces. The theory of set valued maps has applications in control theory, convex optimization, differential inclusions, and economics. The study of fixed points for multivalued contraction mappings using the Hausdorff metric was initiated by Nadler

2. Preliminaries

Definition 2.1[wang,s] A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (t-norm) if the following conditions hold:

(i) T is associative and commutative

(ii) $T(a, 1) = a, a \in [0, 1]$,

(iii) $T(a, b) \leq T(c, d)$, whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$,

Three basic examples of continuous t-norms are(minimum, product and Lukasiewicz t-norm, respectively).

$$T_{min}(a, b) = \min\{a, b\},\ T_P(a, b) = a \cdot b \text{ and } T_L(a, b) = \max\{a + b - 1, 0\}$$

Definition 2.2[schweizer ,B] Let T be a t-norm and $T_n : [0, 1] \to [0, 1]$, $n \in N$, be defined in the following way: $T_1(x) = T(x, x)$, $T_{n+1}(x) = T(T_n(x), x)$, $n \in N$, $x \in [0, 1]$. We say that the T is of H-type if the family $T_n(x) n \in N$ is equi-continuous at $x = 1$. A trivial example of t-norm of H-type is T_{min}.

$$\text{By } T_{i=1}^{0} x_i = 1,\ T_{i=1}^{n} x_i = 1\ x_i = T(T_{i=1}^{n-1} x_i = 1\ x_i, x_n),\ x_1, x_2, \ldots, x_n \in [0, 1],$$

t-norm T could be uniquely extended to an n-ary operation [clement ,E.P.;Mesiar ,R;Pap]. The extension of t-norm T to a countable infinite operation is done as follows:

$$T_{i=1}^{n} x_i = \lim_{n\to\infty} T_{i=1}^{n} = 1\ x_i,\ x_n \in [0, 1],\ n \in N,$$

where $T_{i=1}^{n} = 1\ x_i$ exists since the sequence $\lim_{n\to\infty} T_{i=1}^{n} = 1\ x_i,\ x_n\ [0,1]\ n \in N$ is non-increasing and bounded from below.

$$\text{Let } \lim_{n\to\infty} T_{i=1}^{n} x_i = 1 \text{ and}$$

$$\lim_{n\to\infty} T_{i=1}^{n} x_i = \lim_{n\to\infty} T_{i=1}^{n} x_{n+i} = 1$$

$$\text{Then, } \lim_{n\to\infty} T_{i=1}^{n} x_i = 1$$

$$\text{if and only if, } \sum_{i=1}^{\infty} (1 - x_i) < \infty,$$

$$\text{for } T = T_L \text{ and } T = T_p \text{ while}$$

$$, \lim_{n\to\infty} T_{i=1}^{n} x_i = 1 \text{ if implies, } \sum_{i=1}^{\infty} (1 - x_i) < \infty,$$

For $T \geq T_L$

Proposition 2.3 [Hadzic] Let $\{x_n\} n \in N$ be a sequence of numbers from [0, 1] such that $\lim_{n\to\infty} x_n = 1$ and the t-norm T is of H-type. Then $\lim_{n\to\infty} T_{i=1}^{n} x_i = \lim_{n\to\infty} T_{i=1}^{n} x_{n+i} = 1$

Definition 2.4 (George and Veeramani [9]). A triple (X, M, T) is called a fuzzy metric space if X is a non-empty set, T is a continuous t-norm and $M : X^2 \times (0, \infty) \to (0, 1]$ is a fuzzy set satisfying the following conditions:

(GV1) $M(x, y, t) > 0$,

(GV2) $M(x, y, t) = 1$ if and only if $x = y$,

(GV3) $M(x, y, t) = M(y, x, t)$,

(GV4) $M(x, z, t + s) \geq T(M(x, y, t), M(y, z,s))$,

(GV5) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous, for all $x, y, z \in X$ and $t,s > 0$.

Definition 2.5 ([9]). Let (X, M, T) be a fuzzy metric space. Then, A sequence $\{x_n\}n\in N$

I. converge to $x \in X$ (i.e., $\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} M(x_n, x, t) = 1, t > 0$.
II. Cauchy if, for each $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all $m, n \geq n_0$.
A fuzzy metric (X,M,T) is complete if every cauchy sequence is convergent .

A fuzzy metric space (X, M, T) is complete if every Cauchy sequence is convergent.

Note: a fixed point results in the probabilistic metric spaces with the following generalization of the Banach's contraction principle:

$$F_{Tu,Tv}(qx) \geq \min\{F_{u,v}(x), F_u,T_u(x), F_v,T_v(x), F_u,T_v(2x), F_v,T_u(2x)\}, \qquad (1)$$

where $x > 0$, are studied. Mappings F which, for some $q \in (0, 1)$, satisfies condition (1) are named quasi-contractive mappings. In [11] is used t-norm T such that $T(x, x) \geq x$, $x \in [0, 1]$, which means that $T = T_{min}$. if possibilities for further extensions of t-norm in the context of fixed point problems with quasi-contractive mappings and $\mathcal{H} -$ contractive mappings in the fuzzy metric spaces are elaborated.

Let (X, d) be a metric space and mapping $T : X \to X$. Recently, Kumam et al. [22] presented the following generalization contractive condition (1) of Ciri´c, ´

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx),$$

$$d(T^2 x, x), d(T^2 x, Tx), d(T^2 x, y), d(T^2 x, Ty)\},$$

for all $x, y \in X$ and some $q \in [0, 1)$. In this case, they called the given condition a generalized quasi-contraction.

Definition 2.6(Gregori and Sapena)Let (X, M, T) be a fuzzy metric space. $f : X \to X$ is called a fuzzy contractive mapping if there exists $k \in (0, 1)$ such that

$$\left(\frac{1}{M(f x,f y,t)} - 1\right) \leq k \ 1 \left(\frac{1}{M(x,y,t)} - 1\right), \qquad (2)$$

for each $x, y \in X$ and $t > 0$, k is called the contractive constant of f .

Definition 2.7 (Mihet) Let Ψ be the class of all mappings $\Psi : (0, 1] \rightarrow (0, 1]$ such that Ψ is continuous, non-decreasing and $\Psi(t) > t$ for all $t \in (0, 1)$. Let $\Psi \in \boldsymbol{\Psi}$. A mapping $f : X \rightarrow X$ is said to be fuzzy Ψ -contractive mapping if

$$M(f x, f y, t) \geq \Psi (M(x, y, t)), \qquad (3)$$

for all $x, y \in X$ and $t > 0$.

Definition2.8 (Wardowski) Denoted by $\mathcal{H}$ the family of mappings $\eta : (0, 1] \rightarrow [0, \infty)$ satisfying the following two conditions:

(H1) η transforms (0, 1] onto [0, ∞);

(H2) η is strictly decreasing.

Note that (H1) and (H2) imply that $\eta(1) = 0$.

Definition 2.9 Let (X, M, T) be a fuzzy metric space. A mapping $f : X \rightarrow X$ is said to be fuzzy $\mathcal{H}$ -contractive with respect to $\eta \in \mathcal{H}$ if there exists $k \in (0, 1)$ satisfying the following condition

$$\eta (M(f x, f y, t)) \leq k\eta(M(x, y, t)), \qquad (4)$$

for all $x, y \in X$ and $t > 0$.

Note that for a mapping $\eta \in \mathcal{H}$ of the form $\eta(t) = \frac{1}{t-1}$, $t \in (0, 1]$,

Remark 1. It has been shown in [26] that the class of fuzzy H-contractive mappings are included in the class of ψ-contractive mappings.

Proposition 2.10. Let (X, M, T) be a fuzzy metric space and let $\eta \in H$. A sequence $\{x_n\}n\in N$ in X is

I. Cauchy if and only if, for every $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in N$ such that $\eta (M(x_m, x_n, t)) < \varepsilon$, for all $m, n \geq n_0$
II. convergent to $x \in X$ if and only if, $\lim_{n\to\infty} \eta(M(xn, x, t)) = 0$, for all $t > 0$.

Theorem 2.11 (Wardowski [25]). Let (X, M, T) be a complete fuzzy metric space and let $f : X \rightarrow X$ be a fuzzy H-contractive mapping with respect to $\eta \in H$ such that

(a) $T_{n=1}^{k} x_i\, M(x, f x, t_n) \neq 0$, for all $x \in X$, $k \in N$ and any sequence $\{t_n\}n\in N \subset (0, \infty)$, $t_n \searrow 0$;

(b) $T(r,s) > 0$ implies $\eta(T(r,s)) \leq \eta(r) + \eta(s)$, for all $r,s \in \{M(x, f x, t) : x \in X, t > 0\}$;

(c) $\eta(M(x, f x, t^n) : n \in N\}$ is bounded for all $x \in X$ and any sequence $\{t_n\}n\in N \subset (0, \infty)$, t_n & 0. Then, f has a unique fixed point $x* \in X$ and for each $x_0 \in X$, the sequence $\{ f^{n}x_0\}n\in N$ converges to x^* .

Further, motivated by the contractive condition (1) of Ciri´c, in [´ 27] fuzzy $\mathcal{H}$contractive mappings are generalized and the existence of a fixed point for fuzzy $\mathcal{H}$-quasi-contractive mapping is proven. **Definition 2.12[aminim-harandi] :**Let (X, M, T) be a fuzzy metric space. A mapping $f : X \to X$ is said to be fuzzy H-quasi-contractive with respect to $\eta \in H$ if there exists $k \in (0, 1)$, satisfying the following condition:

$$\eta(M(f x, f y, t)) \le k \max\{\eta(M(x, y, t)), \eta(M(x, f x, t)), \quad (5)$$

$$\eta(M(y, f y, t)), \eta(M(x, f y, t)), \eta(M(y, f x, t))\},$$

for all $x, y \in X$ and any $t > 0$.

In the last part of the next section fuzzy $\mathcal{H}$-quasi-contractive mappings are generalized in the spirit of generalized quasi-contractions [kuman .p.dung] and fixed point result in fuzzy metric spaces is presented. Moreover, the mentioned generalization is confirmed by example.

3.**Main Results**

In this section, we use the fuzzy metric spaces in the sense of Definition 3 with additional condition if $\lim_{n\to\infty} M(x_n, x, t) = 1$, $x, y \in X$.

To prove the results, we use the following very important lemma:

Lemma 3. 1. Let $\{x_n\}$ be a sequence in fuzzy metric space (X, M, T). If there exists $q \in (0, 1)$ such that

$$M(x_n, x_{n+1}, t) \ge M(x_{n-1}, xn, \frac{t}{q}), t > 0, n \in N, \quad (6)$$

$$\text{and } \lim_{n\to\infty} T_{i=n}^{\infty} M(x_0, x_1, t) = 1= 1, \mu \in (0, 1), \quad (7)$$

then $\{x_n\}$ is a Cauchy sequence.

Proof. Let $\sigma \in (q, 1)$ and let $t > 0$. Then $\sum_{i=1}^{\infty} \sigma^i < \infty$; therefore, there exists $n_0 = n_0(t)$, such that

$\sum_{i=1}^{\infty} \sigma^i \le t$. Clearly, condition (6) implies that

$$M(x_n, x_{n+1}, t) \ge M(x_0, x_1, \frac{t}{q^n}), n \in N.$$

For $n \ge n_0$, $m \in N$ we have

$M(x_n, x_{n+m}, t) \ge M(x_n, x_{n+m}, \sum_{i=1}^{\infty} \sigma^i) \ge M(x_n, x_{n+m}, \sum_{i=1}^{n+m-1} \sigma^i)$

$\ge T(T(\ldots T \mid \{z\}_{(m-1)-\text{times}} (M(x_n, x_{n+1}, \sigma^n), \ldots, M(x_{n+m-1}, x_{n+m}, \sigma^{n+m-1})))$

$\ge T(T(\ldots T \mid \{z\} (m-1)-\text{times} (M(x_0, x_1, \frac{\sigma^n}{q^n}), \ldots, M(x_0, x_1, \frac{\sigma^{n+m-1}}{q^{n+m-1}}))).$

Let $\mu= \frac{q}{\sigma} \in (0, 1)$.

Then $M(x_n, x_{n+m}, t) \geq T_{i=n}^{n+m-1} M(x_0, x_1, \frac{1}{\mu^i}) \geq T_{i=n}^{\infty} M(x_0, x_1, \frac{1}{\mu^i})$ $n \geq n_0$, $m \in N$.

Now, by (7) follows Definition 4 (ii) and $\{x_n\}$ is Cauchy sequence.

Our first new result in this section is the following:

Theorem 3.2. Let (X, M, T_{min}) be a complete fuzzy metric space and let $f : X \to X$ be a quasi-contractive mapping such that, for some $q \in (0, \frac{1}{2})$:

$M(f x, f y, t) \geq \min\{M(x, y, \frac{t}{q}), M(f x, x, \frac{t}{q}), M(f y, y, \frac{t}{q}), M(x, f y, \frac{t}{q}), M(f x, y, \frac{t}{q}) M(f x, fx, \frac{t}{q}), M(f y, x, \frac{t}{q})\}$, 8

for all $x, y \in X$ and $t > 0$. Suppose that there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} T_{i=n}^{\infty} M(x_0, fx_0, \frac{1}{\mu^i}) = 1, \mu \in (0, 1), (9)$$

Then, f has unique fixed point.

Proof. Let $x_n = f x_{n-1}$, $n \in N$, where initial $x_0 \in X$ satisfied (9). Then, observe (8) with $x = x_{n-1}$, $y = x_n$:

$M(x_n, x_{n+1}, t) \geq \min\{M(x_{n-1}, x_n, \frac{t}{q}), M(x_n, x_{n-1}, \frac{t}{q}), M(x_{n+1}, x_n, \frac{t}{q}), M(x_{n-1}, x_{n+1}, \frac{t}{q}),$

$M(x_n, x_n, \frac{t}{q}) M(x_n, x_n, \frac{t}{q}) M(x_{n+1}, x_n, \frac{t}{q})\}$

$\geq \min\{M(x_{n-1}, x_n, \frac{t}{q}), M(x_n, x_{n+1}, \frac{t}{q}), \min M(x_{n-1}, x_n, \frac{t}{q}), M(x_{n-1}, x_n, \frac{t}{2q}),$

$M(x_n, x_n, \frac{t}{2q}) M(x_n, x_{n+1}, \frac{t}{2q}) \min M(x_n, x_n, \frac{t}{2q})$ $M(x_{n+1}, x_n, \frac{t}{2q})$

$t > 0$, $n \in N$. If we suppose that

$\min\{M(x_{n-1}, x_n, \frac{t}{2q}), M(x_n, x_{n+1}, \frac{t}{2q})\} = M(x_n, x_{n+1}, \frac{t}{2q})$, then, using the previous calculations, we get the contradiction

$M(x_n, x_{n+1}, t) \geq M(x_n, x_{n+1}, \frac{t}{2q})$,

since $2q < 1$ and $M(x, y, t)$ is increased by t. Thus,

$M(x_n, x_{n+1}, , \frac{t}{2q}) \geq \min\{M(x_{n-1}, x_n, \frac{t}{q})$, for all $n \in N$, and for $q_1 = 2q$, $q_1 \in (0, 1)$:

$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, \frac{t}{2q})$, $t > 0$, $n \in N$.

By Lemma 1, it follows that $\{x_n\}$ is Cauchy sequence. Space (X, M, T_{min}) is complete and there exist $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. If we put $x = x_n$, $y = x^*$ in (8): $M(x_{n+1}, fx^*, t)$ $) \geq$

$\min\{M(x_n, x^*, \frac{t}{q}), M(x_{n+1}, x_n, \frac{t}{q}), M(f x^*, x^*, \frac{t}{q}), M(x_n, f x^*, \frac{t}{q}), M(f x, y, \frac{t}{q})\ M(f x, y, \frac{t}{q})\ M(f x, y, \frac{t}{q})$
}

$n \in N$, $t > 0$, and take $n \to \infty$ then $M(x*, f x*, t) \geq M(x^*, f x^*, \frac{t}{q})$, $t > 0$,

i.e., x^* is the fixed point for f .

Suppose that x^* and y^* are fixed points for f then, by (8):

$M(f x^*, f y^*, t) \geq \min\{M(x^*, y^*, \frac{t}{q}), M(f x^*, x^*, \frac{t}{q}), M(f y^*, y^*, \frac{t}{q}), M(x^*, f y^* \frac{t}{q}), M(f x^*, y^*, \frac{t}{q})$
$M(f x^*, y^*, \frac{t}{q})\ M(f x^*, y^*, \frac{t}{q})\ \}$,

$t > 0$.

Then, $M(x^*, y^*, t) \geq M(x^*, y^*, M(f x^*, y^*, \frac{t}{q})\},)$, $t > 0$, and $x* = y*$.

Example3.3 Let $X = (0, 3)$, $M(x, y, t) = e^{\frac{-|x-y|}{t}}$ T = TP and

$$f(x) = \begin{cases} 3 - x, & x \in (0.1) \\ 1, & x \in [1,3) \end{cases}$$

Case 1. If $x, y \in [1, 2)$, then $M(f x, f y, t) = 1$, $t > 0$ and conditions (11) and (12) are trivially satisfied.

Case 2. If $x \in [1, 2)$ and $y \in (0, 1)$, then, for $q \geq \frac{1}{3}$, we have

$$M(f x, f y, t) = e^{\frac{-|1-y|}{t}} \geq e^{\frac{-3q|1-y|}{t}} = M(f y, y, \frac{t}{q}),\ t > 0.$$

Case 3. Analogously as in the previous case for $q \geq \frac{1}{3}$, we have

$M(f x, f y, t) \geq M(f x, x, q \geq \frac{t}{q})$, $x \in (0, 1)$, $y \in [1, 2)$, $t > 0$.

Case 4. If $x, y \in (0, 1)$, then, for $q \geq \frac{1}{3}$,

$M(f x, f y, t) = e^{\frac{-|x-y|}{t}} \geq e^{\frac{-|1-y|}{t}} \geq e^{\frac{-3q|1-y|}{t}} = M(f y, y, t q)$, $x > y$, $t > 0$, and

$M(f x, f y, t) \geq M(f x, x, t q)$, $x < y$, $t > 0$.

Thus, conditions (11) and (12) are satisfied for all $x, y \in X$, $t > 0$ and follows that $x = 1$ is a unique fixed point for f .

Theorem 3.4. Let (X, M, T) be a complete fuzzy metric space, $T \geq T_P$ and let $f : X \to X$ is a fuzzy generalized quasi-contractive mapping such that for some $q \in (0, 1)$:

M(f x, f y, t) ≥ min{M(x, y, $\frac{t}{q}$), M(f x, x, $\frac{t}{q}$), $\sqrt{M(x, f y, \frac{2t}{q})}$, $\sqrt{M(x, f y, \frac{2t}{q})}$, 11 for all x, y ∈ X and t > 0. Suppose that there exists x_0 ∈ X such that

$$\lim_{n\to\infty} T_{i=n}^{\infty} \text{ i=nM}(x_0, f x_0, \frac{1}{\mu^i}) = 1, \mu \in (0, 1). \quad 12$$

hen, f has a unique fixed point.

Definition 3.5 [17] Denote by H the family of all onto and strictly decreasing mappings η : (0, 1] → [0, ∞). Let (X, M, ∗) be a fuzzy metric space. A mapping T : X → X is said to be fuzzy H-quasi contractive with respect to η ∈ H if there exists k ∈ (0, 1) satisfying

$$\eta(M(T x, T y, t)) \leq k\eta(M(x, y, t)), \forall x, y \in X \ \forall t > 0.$$

For η(t) = $\frac{1}{t}$− 1 one obtains the class of fuzzy contractive mappings introduced by Gregori and Sapena in [5].

If η ∈ H then η(1) = 0 and η is continuous.

Theorem 3.6. [17] Let (X, M, ∗) be an M-complete fuzzy metric space and let T : X → X be a fuzzy H-quasi contractive mapping with respect to η ∈ H such that:

(a) $\prod_{i=1}^{k}$ M(x, T x, ti) ≠0, for all x ∈ X, k ∈ N and any sequence (t_n) ⊆ (0, ∞), t_n ↓ 0;

(b) r ∗s > 0 ⇒ η(r ∗s) ≤ η(r)+η(s), for all r, s ∈ {M(x, T x, t) : x ∈ X, t > 0};

(c) {η(M(x, T x, t_i)) : i ∈ N} is bounded for all x ∈ X and any sequence (t_n) ⊆ (0, ∞), t_n ↓ 0. Then T has a unique fixed point x^* ∈ X and for each x_0 ∈ X the sequence ($T^n x_{0)n\in N}$ converges to x^*.

Theorem 3.7. Let ∗g be a strict t-norm. If (X, M, ∗) is an M-complete fuzzy metric space under a t-norm ∗ ≥ ∗g and T : X → X is a H-contractive mapping with respect to g with the property M(x, T x, 0+) = limt→0+ M(x, T x, t) > 0 for all x ∈ X, then T has a unique fixed point.

Proof. As the proof follows the lines of the proof of Theorem 3.2. in [17], we only sketch it. Let x ∈ X and (x_n)n∈N, x_n = $T^n x$ be the sequence of iterates of x. Then, for all t > 0, n ∈ N,

g(M(x_n, x_{n+1}, t)) ≤ k ng(M(x, T x, t)). Let m, n ∈ N, m < n and t > 0 be given and let {ai} be a strictly decreasing sequence of positive numbers with $\sum_{i=1}^{\infty} a_i$ = 1. Then

M(x_m, x_n, t) ≥ M(x_m, x_n, $\sum_{i=1}^{\infty} a_i\, t$) ≥ ($x_i$, x_{i+1},$\prod_{i=m}^{n-1} a_i\, t$).

≥ $(*_g)\,{}_{i=m}^{n-1}$ =m g(M(x_i , x_{i+1}, a_it))

This implies

g(M(x_m, x_n, t)) ≤$\sum_{i=1}^{\infty} g(M(x_i x_{i+1} a_i t)$ ≤ $\sum_{i=1}^{\infty} k^i$g(M(x, T x, $a_i t$)

$$\leq g(M(x, T\,x, 0+)\sum_{i=1}^{\infty} k^{i}$$

proving that (x_n) is Cauchy. The fact that the limit of (x_n) is the unique fixed point of T can be easily reproduced from the proof of Theorem 3.5. in [17].

Our main theorem is related to the concept of quasi-contraction, initiated by Lj. B. Ciri´c in [1]. We define a fuzzy ´ H-quasi-contractive mapping as follows

Definition 3.8. Let (X, M, ∗) be a fuzzy metric space. A mapping T : X → X is said to be fuzzy H-quasi-contractive with respect to η ∈ H if there exists k ∈ (0, 1) satisfying the following condition:

η(M(T x, T y, t)) ≤ k max{η(M(x, y, t)), η(M(x, T x, t)), η(M(y, T y, t)), η(M(x, T y, t)), } (13)

for all x, y ∈ X and any t > 0.

Theorem 3.9.

Let (X, M, ∗) be an M-complete fuzzy metric space and let T : X → X be a fuzzy H-quasi-contractive mapping with respect to η ∈ H such that

(a) τ ≥ r ∗ s ⇒ η(τ) ≤ η(r) + η(s), for all r, s, τ ∈ {M(T^ix, T^jx, t) : x ∈ X, t > 0, i, j ∈ N};

(b) {η(M(x, T x, t_i)) : i ∈ N} is bounded for all x ∈ X and any sequence {t_n} ⊆ (0, ∞), t_n ↓ 0.

Then T has a unique fixed point x ∗ ∈ X and for each x ∈ X the sequence {T^nx} converges to x^*.

Proof. For A ⊆ X let δ_t(A) = sup{η(M(x, y, t)) : x, y ∈ A} and for each x ∈ X let

O(x, n) = {x, T x, ..., T^nx} and O(x, ∞) = {x, T x, ...}, n ∈ N. Let x ∈ X be arbitrary. Let n ∈ N and let i, j ∈ {1, 2, ..., n}. Then from (13), we obtain

η(M(T^ix, T^jx, t)) = η(M(T T^{i-1} x, T T^{j-1} x, t)) ≤ k max{ η(M(T T^{i-1} x, T T^{j-1} x, t)) η(M(T^{i-1} x, T^ix, t)), η(M(T^{j-1} x, T^jx, t)), η(M(T^{i-1} x, T^j x, t)), }

≤ kδ_t(O(x, n)), and so η(M(T^ix, T^jx, t)) ≤ kδ_t(O(x, n)), i, j ∈ {1, 2, ..., n}, x ∈ X. (14)

Now, if δ_t(O(x, n)) = η(M(T^{i0} x, T^{j0} x, t)) for some i_0, j_0 > 1, then from (14) it follows δ_t(O(x, n)) ≤ kδ_t(O(x, n)), that is, δ_t (O(x, n)) = 0 and thus η(M(T^ix, T^jx, t)) = 0, ∀i, j ≤ n.

Particularly, η(M(x, T x, t)) = 0, which implies M(x, T x, t) = 1. From (GV2) it follows that x = T x, that is, x is a fixed point for T. In the contrary case, δ_t (O(x, n)) = η(M(x, T^lx, t)), (3) for some l ≤ n.

Then, by choosing a strictly decreasing sequence of positive numbers {a_i}

with $\sum_{i=1}^{\infty}$ a_i = 1, from (15), we deduce δ_t (O(x, n)) = η(M(x, T^lx, t)), = η(M(x, T^lx, $\sum_{i=1}^{\infty}$ a_i t)),

≤ η(M(x, T^lx, $\sum_{i=J+1}^{\infty}$ a_i t)), + η(M(x, T^lx, $\sum_{i=1}^{j}$ a_i t)), ∀ j

and so δ_t (O(x, n)) ≤ l $\lim_{j\to\infty}$ *sup* η(M(x, T^lx, $\sum_{i=J+1}^{\infty}$ a_i t)), + η(M(x, T^lx, t)),

≤ $\lim_{j\to\infty}$ *sup* η(M(x, T^lx, $\sum_{i=J+1}^{\infty}$ a_i t)), + kδ_t (O(x, n)) η

Then

δ_t (O(x, n)) ≤ $\frac{1}{1-k}$ $\lim_{j\to\infty}$ *sup* η(M(x, T^lx, $\sum_{i=J+1}^{\infty}$ a_i t)), (16)

Let n, m, n < m be any natural numbers. From (14), we get

η(M(T^nx, T^mx, t)) = η(M(T T^{n-1}x, $T^{m-n+1}T^{n-1}$x, t))

≤ kδ_t(O(T^{n-1}x, m-n+1)) (5)

From (3), there exists k_1 ≤ m − n + 1 such that

δ_t(O(T^{n-1} x, m − n + 1)) = η(M(T^{n-1} x, T^{k_1} T^{n-1} x, t)).(17)

From (14),(16) and (17), we get

η(M(T^nx, T^mx, t)) = kη(M(T^{n-1}x, T^{k_1} T^{n-1}x, t))

= kη(M(T^{n-2}x, $k_1 + 1$))

≤ k^2 δ_t(O(T n−2x, k1 + 1)) ≤ k_2 δ_t (O(T^{n-2} x, m − n + 2)).

Proceeding in this manner, we obtain η(M(T^nx, T^mx, t)) ≤ k^n δ_t (O(x, m)). (16) From (14) and (17)

it follows η(M(T^nx, T^mx, t)) ≤ $\frac{k^n}{1-k}$ $\lim_{j\to\infty}$ *sup* η(M(x, T^lx, $\sum_{i=J+1}^{\infty}$ a_i t)), (18)

From (18) and (b), we have $\lim_{m,n\to\infty}$ η(M(T^nx, T^mx, t)) = 0, and so $\lim_{m,n\to\infty}$ η(M(T^nx, T^mx, t)) = 1.

Thus, (x_n)n∈N, x_n = T^nx is a Cauchy sequence. By the completeness of X there exists x ∗ ∈ X such that $\lim_{,n\to\infty}$ x_n = x ∗ .

Let t > 0 be given. Then, for each > 0 and n ∈ N, we have M(x^* , T x^* , t + ∈) ≥ M(x^* , T^{n+1} x^* , ∈)) ∗ M(T x^* , T^{n+1} x^* , t) and hence

η M(x^* , T x^* , t + ∈) ≤ η M(x^* , T^{n+1} x^* , ∈)) + η M(x^* , T^{m+1} x^* , t))

≤ η(M(x ∗ , T n+1x ∗ ,)) + k max{η(M(x ∗ , T nx ∗ , t)), η(M(x ∗ , T x∗ , t)), η(M(T^n x ∗ , T^{n+1} x ∗ , t)), η(M(x ∗ , T^{m+1} x ∗ , t)), } Letting n → ∞ (having in mind Lemma 3.3) we obtain

$\eta(M(x^*, Tx^*, t+)) \leq k\eta(M(x*, Tx*, t))$,

and so

$\eta(M(x^*, Tx^*, t)) = \lim_{,\epsilon \to 0^+} \eta M(x*, Tx*, t+\epsilon)), \leq k\eta((M(x^*, Tx^*, t))$

Thus $\eta(M(x^*, Tx^*, t)) = 0$, implying $\eta(M(x^*, Tx^*, t)) = 1$.

To show the uniqueness assume that y^* is a fixed point of T. Then, for all t > 0,

$\eta(M(x^*, Tx^*, t)) = \eta(M(Tx^*, Ty^*, t)) \leq k \max\{\eta(M(x^*, y^*, t)), \eta(M(x^*, Tx^*, t))\ \eta(M(y^*, Ty^*, t)), \eta(M(x^*, Ty^*, t)), \} = k\eta(M(x^*, y^*, t))$. This gives $(M(x^*, y^*, t)). = 1$, that is, $x^* = y^*$.

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