On some abstract differential equations of fractional order with application

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Abstract.
In this paper, we consider the abstract differential equation of fractional order of the form

$$\frac{d^{n-1}}{dt^{n-1}}\left(\frac{d^{\alpha}u}{dt^{\alpha}}-Au\right)=g\left(t,W\right) \text{ with } 0<\alpha\leq 1,$$

we examine the existence and uniqueness of the solution of the initial value problem in a Banach Space E for the considered abstract nonlinear differential equation. We study also the correct formulation to this problem.
An application is obtained for some partial differential equations of fractional order.

1. Introduction

Assume E is a Banach Space. Let $\{(H_i(t), i=1,2,3,\dots,\nu), t\in I=[0,Q_0]\}$ be families of closed linear operators defined on dense sets $S_1, S_2, \dots, S_\nu$ respectively to E.

Let A be a closed linear operator defined on a dense set S in E so that $S\subset E, S\subset S_i, (i=1,2,3,\dots,\nu)$. Assuming that the range of these operators is in E, Consider the abstract nonlinear differential equation with fractional order

$$\frac{d^{n-1}}{dt^{n-1}}\left(\frac{d^{\alpha}u}{dt^{\alpha}}-Au\right)=g\left(t,W\right), \qquad 1.1$$

$$u\big|_{t=0}=h_0, \left.\frac{d^j u}{dt^j}\right|_{t=0}=h_j,\ j=1,2,\dots,n-1. \qquad 1.2$$

Where all the elements $h_0, h_1, h_2, \dots, h_{n-1}\in S$, $W=\left(H_1(t)u, H_2(t)u, \dots, H_\nu(t)u\right)$ and g is a given abstract nonlinear function defined on $I\times E^{\nu}$ with values in E. We can assume, without losing generality, that

$$u\big|_{t=0}=\left.\frac{d^j u}{dt^j}\right|_{t=0}=\vartheta,\ j=1,2,\dots,n-1. \qquad 1.3$$

Where ϑ is the zero element of the Banach space E. Let g be uniformly Hölder continuous for all $t\in I$, satisfies that

$$\left\| g(t,W) - g(t^*,W) \right\| \le K \left| t - t^* \right|^{\beta}, \qquad 1.4$$

For all t and t^* in I and all W in E^ν, the constant $K > 0, 0 < \beta < 1$ and $\| . \|$ is the norm in E.

For all $W, W^* \in E^\nu, W = (w_1, w_2, \dots, w_\nu), W^* = (w^*_1, w^*_2, \dots, w^*_\nu)$, the function g satisfies Lipschitz Condition at $K_1 > 0$,

$$\left\| g(t,W) - g(t,W^*) \right\| \le K_1 \sum_{i=1}^{\nu} \left\| w_i - w_i^* \right\|. \qquad 1.5$$

For each and every $z \in S_i$ the functions $H_i(t)z$ and $H(t)z$ are uniformly Hölder continuous for $t \in I$ and i=1,2,3,…,ν with exponents β' and β'', respectively—Without sacrificing generality, we can assume that $\beta = \beta' = \beta''$. The space of continuous functions $u(t)$ with $t \in I$ and $u(t) \in E$ is symbolize by $C^E(I)$. This space's norm is determined by

$$\|u\|_{C^E(I)} = \max_{t \in I} \|u(t)\|. \qquad 1.6$$

Suppose that A generates a semigroup $\{Q(t), t \in I\}$ strongly continuous $\forall t \ge 0$, this case of semigroup is called C_0 (see[1, 2, 3]). Likewise, suppose that $Q(t)\upsilon \in S, \forall \upsilon \in E, t > 0$ see([4]).

Assume that if there exist $0 < \delta < 1$ and a positive constant M, $\forall \upsilon \in E, t_2 \in I, t_1 \in (0, Q_0]$ with i=1,2,3,…,ν. Then

$$\|H(t_2)Q(t_1)\upsilon\| \le \frac{M}{t_1^\delta}\|\upsilon\|, \qquad 1.7$$

$$\|H_i(t_2)Q(t_1)\upsilon\| \le \frac{M}{t_1^\delta}\|\upsilon\|, \qquad 1.8$$

We prove the existence and uniqueness of the Cauchy Problem solution in this paper see([9]). The correct solution of the problem is also proved, and we conclude with an application of the theory of partial differential equations of fractional order.

2. The solution of the problem

In this part, we discuss the existence and uniqueness of the solution of the initial value problem. Define on $C^E(I)$, a distance function (metric) ρ by

$$\rho(u_1, u_2) = \max_{t \in I} e^{-\lambda t} \|u_1(t) - u_2(t)\|, \qquad 2.1$$

Where $u_1, u_2 \in C^E(I), \lambda > 1$ being a fixed number. $(C^E(I), \rho)$ is clearly a metric space see([1]).

THEOREM 2.1. The abstract initial value problem **(1.1)** has a week solution in the metric space $(C^E(I),\rho),\ \forall t\in I.$

PROOF. From equation **(1.1)**, let

$$\frac{d^\alpha u}{dt^\alpha}-Au=\upsilon,\quad where\ 0<\alpha\le 1. \qquad 2.2$$

The above equation's intended answer u, can be expressed in the form (see[5, 6, 7, 8, 10, 16]).

$$u(t)=\alpha\int_0^t\int_0^\infty \theta\left(t-\eta\right)^{\alpha-1}\zeta_\alpha(\theta)\,Q((t-\eta)^\alpha\theta)\,\upsilon(\eta)\,d\theta\,d\eta, \qquad 2.3$$

Where $\zeta_\alpha(\theta)$ is a probability density function defined on $(0,\infty)$.

Where υ satisfies

$$\frac{d^{n-1}\upsilon}{dt^{n-1}}=g\left(t,W\right), \qquad 2.4$$

Integration **(**2.4**)** $(n-1)$ times (see[1])**,** we get

$$\upsilon(t)=\frac{1}{\Gamma(n)}\int_0^t(t-s)^{n-1}g\left(s,W\right)ds, \qquad 2.5$$

Let T is an operator defined on $C^E(I)$ by

$$T\upsilon(t)=\frac{1}{\Gamma(n)}\int_0^t(t-s)^{n-1}g\left(s,W\right)ds, \qquad 2.6$$

We prove that T is contraction mapping see([7, 9]). We've noticed

$$\left\|T\upsilon-T\upsilon^*\right\|\le\frac{K_1}{\Gamma(n)}\sum_{i=1}^{\nu}\int_0^t(t-s)^{n-1}\left\|w_i-w_i^*\right\|ds, \qquad 2.7$$

$$\left\|T\upsilon-T\upsilon^*\right\|\le\frac{K_1M\alpha}{\Gamma(n)}\int_0^t\int_0^s\int_0^\infty\theta^{1-\delta}\zeta_\alpha(\theta)(t-s)^{n-1}(s-\eta)^{\alpha(1-\delta)-1}\left\|\upsilon-\upsilon^*\right\|d\theta d\eta ds, \qquad 2.8$$

$$\left\|T\upsilon-T\upsilon^*\right\|\le\frac{K_1M\alpha\ \Gamma(\alpha(1-\delta))}{\Gamma(n+\alpha(1-\delta))}\int_0^t\int_0^\infty\theta^{1-\delta}\zeta_\alpha(\theta)(t-\eta)^{n+\alpha(1-\delta)-2}\left\|\upsilon-\upsilon^*\right\|d\theta d\eta, \qquad 2.9$$

Let $\gamma=\alpha(1-\delta)$,

$$\left\|T\upsilon - T\upsilon^*\right\| \le \frac{K_1 M\alpha\,\Gamma(\gamma)}{\Gamma(n+\gamma)}\int_0^t\int_0^\infty \theta^{1-\delta}\zeta_\alpha(\theta)(t-\eta)^{n+\gamma-2}\left\|\upsilon-\upsilon^*\right\|d\theta d\eta, \qquad 2.10$$

We can find that $\int_0^\infty \theta^{1-\delta}\zeta_\alpha(\theta)d\theta \le K_2, where\ K_2$ is constant.

$$\left\|T\upsilon - T\upsilon^*\right\| \le \frac{K_1K_2 M\alpha\,\Gamma(\gamma)}{\Gamma(n+\gamma)}\int_0^t (t-\eta)^{n+\gamma-2}\left\|\upsilon-\upsilon^*\right\|d\eta, \qquad 2.11$$

Assume that $M_* = \frac{K_1K_2M\alpha\,\Gamma(\gamma)}{\Gamma(n+\gamma)}$. Then

$$\left\|T\upsilon - T\upsilon^*\right\| \le M_*\int_0^t (t-\eta)^{n+\gamma-2}\left\|\upsilon-\upsilon^*\right\|d\eta, \qquad 2.12$$

if $\xi = \max_{t\in I} e^{-\lambda t}\left\|\upsilon-\upsilon^*\right\|$,

$$\left\|T\upsilon - T\upsilon^*\right\| \le M_*\int_0^t (t-\eta)^{n+\gamma-2}e^{-\lambda\eta}e^{\lambda\eta}\left\|\upsilon-\upsilon^*\right\|d\eta, \qquad 2.13$$

Therefore see([14, 6, 20])

$$\left\|T\upsilon - T\upsilon^*\right\| \le M_*\xi\int_0^t (t-\eta)^{n+\gamma-2}e^{\lambda\eta}d\eta, \qquad 2.14$$

$$\left\|T\upsilon - T\upsilon^*\right\| \le M_*\xi\left[\frac{1}{\lambda^{n+\gamma}}\int_0^{t-\frac{1}{\lambda}} e^{\lambda\eta}d\eta + \int_{t-\frac{1}{\lambda}}^{t}(t-\eta)^{n+\gamma-2}e^{\lambda\eta}d\eta\right], \qquad 2.15$$

$$\left\|T\upsilon - T\upsilon^*\right\| \le M_*\xi\left[\left(\frac{1}{\lambda}\right)^{n+\gamma-1}(1+\frac{1}{n+\gamma-1})e^{\lambda t}\right], \qquad 2.16$$

$$e^{-\lambda t}\left\|T\upsilon - T\upsilon^*\right\| \le \left(\frac{M_*\xi}{\lambda^{n+\gamma-1}}\right)(1+\frac{1}{n+\gamma-1}), \qquad 2.17$$

let $M^n = \left(\frac{(n+\gamma)M_*}{(n+\gamma-1)}\right)$, then

$$e^{-\lambda t}\left\|T\upsilon - T\upsilon^*\right\| \le \left(\frac{M^n}{\lambda^{n+\gamma-1}}\right)\xi, \qquad 2.18$$

which gives

$$\rho(T\upsilon, T\upsilon^*) \le \left(\frac{M^n}{\lambda^{n+\gamma-1}}\right)\rho(\upsilon, \upsilon^*). \qquad 2.19$$

For a sufficiently large λ we deduce that T is a contraction operator therefore there exists a unique fixed point such that $T\upsilon = \upsilon \in C^E(I)$, which proves the existence and uniqueness of a weak solution u in $C^E(I)$.

Remark 2.1. If we take $\zeta_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\,\Phi_\alpha(\theta^{-1/\alpha}) \ge 0$ is the probability density function defined on $(0,\infty)$.

It is not hard to prove that $\chi \in [0,1]$.

$$\int_0^\infty \theta^\chi \,\zeta_\alpha(\theta)\,d\theta = \int_0^\infty \theta^{-\alpha\chi}\,\Phi_\alpha(\theta)\,d\theta = \frac{\Gamma(1+\chi)}{\Gamma(1+\chi\alpha)}. \qquad 2.20$$

Where $\Phi_\alpha(\theta)$ is one-sided stable probability density (see[11, 12, 13, 15]), The proof of our key results will be based on the following Theorems.

We will prove that $\left|g\left(t, W(t)\right)\right|$ is bounded on the interval $[0, Q_0]$.

Theorem 2.2. $\left|g\left(t, W(t)\right)\right|$ is bounded $\forall t \in I$, if the function $g\left(t, W(t)\right)$ satisfies the condition (1.4) and (1.8).

PROOF. From condition (1.4), it is clear that

$$\begin{aligned} &\left\|g\left(t, W(t)\right) - g\left(t, \vartheta, \ldots, \vartheta\right)\right\| = \left\|g\left(t, W(t)\right) - g\left(t, W(0)\right)\right\| \le k\sum_{i=1}^{\nu}\|w_i\| \\ &= k\alpha\sum_{i=1}^{\nu}\left\|\int_0^t\int_0^\infty \theta\,\left(t-\eta\right)^{\alpha-1}\,\zeta_\alpha(\theta)\,H_i(t)\;Q((t-\eta)^\alpha\theta)\;\upsilon(\eta)\,d\theta\,d\eta\right\|. \end{aligned} \qquad 2.21$$

From **Remark 2.1.** direct calculation gives that

$$\int_0^\infty \theta\,\zeta_\alpha(\theta)\,d\theta = \frac{1}{\Gamma(1+\alpha)}, \qquad 2.22$$

at $\chi = 1$, Then from (1.8), we get the required result.

Theorem 2.3. (see[1, 2]) The function $u(t)$ is an element of S, $\forall t \in I$ and so $u \in C^S[0, Q_0]$.

PROOF. It is enough to proof that $\upsilon(t)$ satisfies the Lipschitz Condition.

Let $t_2 > t_1$,

$$\upsilon(t_2)-\upsilon(t_1)=\frac{1}{\Gamma(n)}\int_0^{t_2}(t_2-s)^{n-1}g\left(s,W\right)ds-\frac{1}{\Gamma(n)}\int_0^{t_1}(t_1-s)^{n-1}g\left(s,W\right)ds, \qquad 2.23$$

$$\upsilon(t_2)-\upsilon(t_1)=\frac{1}{\Gamma(n)}\int_0^{t_1}\left[(t_2-s)^{n-1}-(t_1-s)^{n-1}\right]g\left(s,W\right)ds+\frac{1}{\Gamma(n)}\int_{t_1}^{t_2}\left[(t_2-s)^{n-1}\right]g\left(s,W\right)ds, \qquad 2.24$$

$$\left|\upsilon(t_2)-\upsilon(t_1)\right|\le\frac{1}{\Gamma(n)}\int_0^{t_1}\left[(t_2-s)^{n-1}-(t_1-s)^{n-1}\right]\left|g\left(s,W\right)\right|ds+\frac{1}{\Gamma(n)}\int_{t_1}^{t_2}\left[(t_2-s)^{n-1}\right]\left|g\left(s,W\right)\right|ds, \qquad 2.25$$

And we know that $\left|g\left(t,W(t)\right)\right|$ is bounded function.

It's clear that

$$\left|\upsilon(t_2)-\upsilon(t_1)\right|\le K^{**}\left|t_2-t_1\right|, \qquad 2.26$$

Then $\upsilon(t)$ is satisfies the Lipschitz condition with $K^{**}>0$.

To finish the theorem of the existence and uniqueness of the arrangement (strongly) we demonstrate that every one of differentiation.

$$\frac{du}{dt},\frac{d^2u}{dt^2},\ldots,\frac{d^{n-1}u}{dt^{n-1}}, \qquad 2.27$$

be included in $C^S\left(I\right)$, but $\mu_1(t)=g\left(t,W\left(t\right)\right)$. Then formally we can write

$$\frac{d^r u\left(t\right)}{dt^r}=\frac{\alpha}{\Gamma\left(n-r\right)}\int_0^t\int_0^{t-\xi}\int_0^\infty \theta\left(\xi\right)^{\alpha-1}\left(\left(t-\xi\right)-s\right)^{n-r-1}\zeta_\alpha(\theta)Q((\xi)^\alpha\theta)\,\mu_1\left(s\right)d\theta\,ds\,d\xi, \qquad 2.28$$

to obtain the expected outcome, we need to show that μ_1 fulfills a uniform Hölder

Condition for $t\in I$, we can also, let $\mu_2(t)=H\left(t\right)u\left(t\right)$ and show that $\mu_2\left(t\right)$ fulfills a uniform Hölder Condition too. Which

$$\mu_2\left(t\right)=\alpha\int_0^t\int_0^\infty\theta\left(t-\eta\right)^{\alpha-1}\zeta_\alpha(\theta)\,H\left(t\right)Q((t-\eta)^\alpha\theta)\,\upsilon(\eta)\,d\theta\,d\eta. \qquad 2.29$$

Suppose that $t_2>t_1$, As a result, it's simple to prove that

$$\begin{aligned}\mu_2(t_2)-\mu_2(t_1) &= \alpha\int_{t_1}^{t_2}\int_0^{\infty}\theta(t_2-\eta)^{\alpha-1}\zeta_\alpha(\theta)H(t_2)Q((t_2-\eta)^\alpha\theta)\upsilon(\eta)\,d\theta\,d\eta \\ &+\alpha\int_0^{t_1}\int_0^{\infty}\theta\left[(t_2-\eta)^{\alpha-1}-(t_1-\eta)^{\alpha-1}\right]\zeta_\alpha(\theta)H(t_2)Q((t_2-\eta)^\alpha\theta)\upsilon(\eta)\,d\theta\,d\eta \\ &+\alpha\int_0^{t_1}\int_0^{\infty}\theta\left[(t_1-\eta)^{\alpha-1}\right]\zeta_\alpha(\theta)\left[H(t_2)-H(t_1)\right]Q((t_2-\eta)^\alpha\theta)\upsilon(\eta)\,d\theta\,d\eta \\ &+\alpha\int_0^{t_1}\int_0^{\infty}\theta\left[(t_1-\eta)^{\alpha-1}\right]\zeta_\alpha(\theta)\,H(t_1)\left[Q((t_2-\eta)^\alpha\theta)-Q((t_1-\eta)^\alpha\theta)\right]\upsilon(\eta)\,d\theta\,d\eta.\end{aligned} \quad 2.30$$

We can see that, according to (**[5]**)

$$\left\|\mu_2(t_2)-\mu_2(t_1)\right\| \le N\left[(t_2-t_1)^\gamma\right]+\frac{N}{\alpha}(t_2-t_1)^\gamma+N\gamma\delta_2^{\gamma-1}(1-c)^{-\gamma-1}(t_2-t_1)^\gamma, \quad 2.31$$

which N is positive constant depended on $\alpha,\delta,\sup_j\left\|\upsilon(\eta)\right\|$ *and* $0<\delta_2<1$, Thus $\mu_2(t)$ satisfies a uniform Hölder condition on I.

$$\begin{aligned}&\left\|\mu_1(t_2)-\mu_1(t_1)\right\|\le\left\|g(t_2,W(t_2))-g(t_1,W(t_2))\right\|+\left\|g(t_1,W(t_2))-g(t_1,W(t_1))\right\| \\ &\le k(t_2-t_1)^\beta+K_1\sum_{i=1}^{\nu}\left\|H_i(t_2)u(t_2)-H_i(t_1)u(t_1)\right\|,\end{aligned} \quad 2.32$$

where k *and* k_1 are positive constants. Then $\mu_1(t)$ *and* $\mu_2(t)$ satisfies Hölder Condition for $t\in I$, therefore $\left({du}/{dt}\right)\in C^s(I)$ *and* $\left({d\upsilon}/{dt}\right)$ is continuous $\forall t\in I$.

Now, $Au(t)$ can be written in the form

$$Au(t)=\frac{\alpha}{\Gamma(n)}\int_0^t\int_0^\eta\int_0^\infty\theta(t-\eta)^{\alpha-1}\,(t-s)^{n-1}\zeta_\alpha(\theta)AQ((t-\eta)^\alpha\theta)\,\mu_1(s)\,d\theta\,ds\,d\eta, \quad 2.33$$

thus differentiate $(n-1)$ times we get

$$\frac{d^{n-1}}{dt^{n-1}}\left[Au(t)\right]=\alpha\int_0^t\int_0^\eta\int_0^\infty\theta(t-\eta)^{\alpha-1}\,\zeta_\alpha(\theta)AQ((t-\eta)^\alpha\theta)\,\mu_1(s)\,d\theta\,ds\,d\eta=A\frac{d^{n-1}u(t)}{dt^{n-1}}. \quad 2.34$$

therefore,

$$\frac{d^{n-1}}{dt^{n-1}}\left(\frac{d^\alpha u(t)}{dt^\alpha}\right)=\frac{d^{n-1}\upsilon(t)}{dt^{n-1}}+A\frac{d^{n-1}u(t)}{dt^{n-1}}, \quad 2.35$$

is continuous on I. consequently,

$$u(t)=\alpha\int_0^t\int_0^\infty \theta\left(t-\eta\right)^{\alpha-1}\zeta_\alpha(\theta)\,Q((t-\eta)^\alpha\theta)\,\upsilon(\eta)\,d\theta\,d\eta, \quad 2.36$$

represent the unique solution of the considered Cauchy Problem.

3. **The correct formulation**

In this section, we prove the correct formulation of the considered initial value problem (1.1) and (1.2). In other words, we prove the continuous dependent of the solution of the problem on the initial conditions. Let $\{u^{\mathrm{M}}\}$ be a sequence of solutions of the initial value problem

$$\frac{d^{n-1}}{dt^{n-1}}\left(\frac{d^\alpha u^{\mathrm{M}}}{dt^\alpha}-Au^{\mathrm{M}}\right)=g\left(t,W^{\mathrm{M}}\right), \quad 3.1$$

$$u^M\Big|_{t=0}=h_0^M\in S,\left.\frac{d^j u^M}{dt^j}\right|_{t=0}=h_j^M,\ j=1,2,\dots,n-1. \quad 3.2$$

Where W^{M} is the sequence $W^{\mathrm{M}}=\left(H_1(t)u^{\mathrm{M}},H_2(t)u^{\mathrm{M}},\dots,H_\nu(t)u^{\mathrm{M}}\right)$.

Theorem 3.1. Let the sequences $\left\{h_0^{\mathrm{M}}\right\},\left\{h_1^{\mathrm{M}}\right\},\left\{h_2^{\mathrm{M}}\right\},\dots,\left\{h_{n-1}^{\mathrm{M}}\right\},\left\{Ah_{n-1}^{\mathrm{M}}\right\}$, be convergent in E to $h_0,h_1,h_2,\dots,h_{n-1},Ah_{n-1}$, respectively. If the sequence $\left\{H(t)h_0^{\mathrm{M}}\right\},\left\{H(t)h_1^{\mathrm{M}}\right\},\left\{H(t)h_2^{\mathrm{M}}\right\},\dots,\left\{H(t)h_{n-1}^{\mathrm{M}}\right\}$ $,\left\{H_i(t)h_0^{\mathrm{M}}\right\},\left\{H_i(t)h_1^{\mathrm{M}}\right\},\left\{H_i(t)h_2^{\mathrm{M}}\right\},\dots,\left\{H_i(t)h_{n-1}^{\mathrm{M}}\right\}$, are uniformly convergent with respect $t\in I$ in E to $H(t)h_0,H(t)h_1,H(t)h_2,\dots,H(t)h_{n-1},H_i(t)h_0,H_i(t)h_1,H_i(t)h_2,\dots,H_i(t)h_{n-1}$, $i=1,2,3,\dots\dots,\nu$, respectively, where $h_0,h_1,h_2,\dots,h_{n-1}$ are elements in G, then the sequence of the solutions $\{u^{\mathrm{M}}(t)\}$ of the problem 3.1 and 3.2 converges in the metric space $C^E(I)$ to the solution $u(t)$ of the considered problem with the initial condition (see[1- 18])

$$u\big|_{t=0}=h_0,\left.\frac{d^j u}{dt^j}\right|_{t=0}=h_j,\ j=1,2,\dots,n-1. \quad 3.3$$

PROOF. Let

$$\omega^{\mathrm{M}}(t)=u^{\mathrm{M}}(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}h_k^{\mathrm{M}}, \quad 3.4$$

substitute in the equation 1.1, we get

$$\frac{d^{n-1}}{dt^{n-1}}\left(\frac{d^\alpha\omega^{\mathrm{M}}(t)}{dt^\alpha}-A\omega^{\mathrm{M}}(t)\right)=g^*\left(t,W^{\mathrm{M}}\right), \quad 3.5$$

with the initial conditions

$$\omega^{\mathrm{M}}\Big|_{t=0}=0,\left.\frac{d^j\omega^{\mathrm{M}}}{dt^j}\right|_{t=0}=0,\ j=1,2,\dots,n-1. \quad 3.6$$

Take

$$\frac{d^{\alpha}\omega^{M}(t)}{dt^{\alpha}}-A\omega^{M}(t)=\frac{d^{\alpha}}{dt^{\alpha}}\left[u^{M}(t)-\sum_{k=0}^{n-1}\frac{t^{k}}{k!}h_{k}^{M}\right]-A\left[u^{M}(t)-\sum_{k=0}^{n-1}\frac{t^{k}}{k!}h_{k}^{M}\right]=$$
$$\frac{d^{\alpha}u^{M}(t)}{dt^{\alpha}}-Au^{M}(t)+A\sum_{k=0}^{n-1}\frac{t^{k}}{k!}h_{k}^{M}-\left[\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(1-\alpha+k)}h_{k}^{M}\right], \quad 3.7$$

substitute in the equation (3.5),

$$\frac{d^{n-1}}{dt^{n-1}}\left(\frac{d^{\alpha}u^{M}(t)}{dt^{\alpha}}-Au^{M}(t)\right)-\frac{d^{n-1}}{dt^{n-1}}\left(\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(1-\alpha+k)}h_{k}^{M}\right)+A\,\frac{d^{n-1}}{dt^{n-1}}\left(\sum_{k=0}^{n-1}\frac{t^{k}}{k!}h_{k}^{M}\right)=g^{*}\left(t,W^{M}\right), \quad 3.8$$

where

$$g\left(t,W^{M}\right)-\frac{d^{n-1}}{dt^{n-1}}\left(\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(1-\alpha+k)}h_{k}^{M}\right)+A\,\frac{d^{n-1}}{dt^{n-1}}\left(\sum_{k=0}^{n-1}\frac{t^{k}}{k!}h_{k}^{M}\right)=g^{*}\left(t,W^{M}\right), \quad 3.9$$

then we can write in the form,

$$g\left(t,W^{M}\right)-\frac{d^{n-1}}{dt^{n-1}}\left(\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(1-\alpha+k)}\,h_{k}^{M}\right)+A\,h_{n-1}^{M}=g^{*}\left(t,W^{M}\right). \quad 3.10$$

Set

$$\frac{d^{\alpha}\omega^{M}(t)}{dt^{\alpha}}-A\omega^{M}(t)=\varpi^{M}(t), \quad 3.11$$

thusly,

$$\frac{d^{n-1}}{dt^{n-1}}\left(\varpi^{M}(t)\right)=g^{*}\left(t,W^{M}\right), \quad 3.12$$

It’s easy to see that

$$\varpi^{M}(t)=\int_{0}^{t}\int_{0}^{\xi_{n-1}}\ldots\int_{0}^{\xi_{3}}\int_{0}^{\xi_{2}}g^{*}\left(\xi,W^{M}(\xi)\right)d\xi_{1}\,d\xi_{2}\ldots d\xi_{n-1}, \quad 3.13$$

we can find that

$$\left\|\varpi^{M}(t)-\varpi^{R}(t)\right\|\le\int_{0}^{t}\int_{0}^{\xi_{n-1}}\ldots\int_{0}^{\xi_{3}}\int_{0}^{\xi_{2}}\left\|g^{*}\left(\xi,W^{M}(\xi)\right)-g^{*}\left(\xi,W^{R}(\xi)\right)\right\|d\xi_{1}\,d\xi_{2}\ldots d\xi_{n-1}, \quad 3.14$$

$$\left\|\varpi^{M}(t)-\varpi^{R}(t)\right\|\le\int_{0}^{t}\int_{0}^{\xi_{n-1}}\ldots\int_{0}^{\xi_{3}}\int_{0}^{\xi_{2}}[\left\|g\left(t,W^{M}\right)-g\left(t,W^{R}\right)\right\|$$
$$+A\left\|h_{n-1}^{M}-h_{n-1}^{R}\right\|+\sum_{k=0}^{n-1}\frac{t^{k-\alpha-n+1}}{\Gamma(k-\alpha-n+2)}\left\|h_{k}^{M}-h_{k}^{R}\right\|]\,d\xi_{1}\,d\xi_{2}\ldots d\xi_{n-1}, \quad 3.15$$

multiply by $e^{-\lambda t}$ and using the metric defined by equation (2.1), we get

$$\rho\left(\varpi^{\mathrm{M}},\varpi^{R}\right)\leq K\,\rho\left(Ah_{n-1}^{\mathrm{M}},Ah_{n-1}^{R}\right)+k\left[\sum_{k=0}^{n-1}\rho\left(h_k^{\mathrm{M}},h_k^{R}\right)\right]+k\left[\sum_{k=1}^{\nu}\sum_{j=0}^{n-1}\rho\left(A_k\,h_j^{\mathrm{M}},A_k\,h_j^{R}\right)\right]. \qquad 3.16$$

As per every one of the circumstances previously, the sequence $\left\{\varpi^{\mathrm{M}}\right\}$ is fundamental and hence converges to ϖ in $C^{E}(I)$. But

$$\omega^{\mathrm{M}}(t)=\alpha\int_0^t\int_0^\infty\theta\left(t-\eta\right)^{\alpha-1}\zeta_\alpha(\theta)Q((t-\eta)^{\alpha}\theta)\varpi^{\mathrm{M}}(\eta)\,d\theta\,d\eta. \qquad 3.17$$

Therefore, the sequence $\{u^{\mathrm{M}}(t)\}$ uniformly converges with respect to $t\in I$ in E to required solution.

4. Application

In this section, we present application of the proposed method in an initial condition problem. The existence and the uniqueness of the solution are also introduced. We consider the Cauchy Problem in $\mathbb{R}^n$ for a class of semi-linear parabolic partial differential equations. We assume that the coefficients locally Lipschitz, not necessarily differentiable, with continuous data and local uniform ellipticity (see[17, 18, 19, 20]).

Let $Z\subset\mathbb{R}^n$ be an open set with boundary ∂Z and its closure is $\bar{Z}$ and let $L_2(Z)$ be the set of all square integrable functions on Z.

Assume that $C_0^k(Z)\subset C^k(Z)$ be the set of all functions having a compact support, where $C^k(Z)$ defined as the set of all continuous real-valued functions and Let $B^k(Z)$ be the complete space of $C^k(Z)$ with respect to norm.

$$\|g\|_k^2=\left[\sum_{|p|\leq k}\int_Z\left|D^p g(x)\right|^2dx\right]. \qquad 4.1$$

Where $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, $p=(p_1,p_2,\dots,p_n)$ is an6n-dimensional,multi-index, $|p|=p_1+p_2+\dots+p_n$, $D_i=\dfrac{\partial}{\partial x_i}$ and $D^p=D_1^{p_1}D_2^{p_2}\dots D_n^{p_n}$.

Firstly, let L be differential operator

$$Lu(x,t)=\frac{\partial^\alpha u}{\partial t^\alpha}+A(x,D)u, \qquad 4.2$$

where

$$A(x,D)=\sum_{|p|\leq 2k}a_p(x)D^p. \qquad 4.3$$

Then the Cauchy Problem is

$$\frac{\partial^{n-1}}{\partial t^{n-1}}\left(Lu\left(x,t\right)\right)=\sum_{|p|\leq 2k-1} b_p\left(x,t\right)D^p u+g\left(x,t,W\right), \qquad 4.4$$

with the initial conditions

$$\frac{\partial^{j-1}u\left(x,0\right)}{\partial t^{j-1}}=h_{j-1}, \forall x\in\mathbb{R}^n, j=1,2,3,\ldots,n-1, \qquad 4.5$$

and

$$W=\left(H_1(t)u,H_2(t)u,\ldots\ldots\ldots,H_\nu(t)u\right),\quad H_i\left(t\right)=\sum_{|p|\leq 2k-1} C_{p,i}\left(x,t\right)D^p. \qquad 4.6$$

Let $Lu\left(x,t\right)$ such that, all the coefficient a_p are continuous on $\bar{Z}$ and if

$$\left(-1\right)^{k-1}\sum_{|p|=2m} a_p\left(x\right)\varepsilon^p \geq C\left|\varepsilon\right|^{2k},\ C>0, \qquad 4.7$$

then L is called is uniformly parabolic in $\bar{Z}$.

Lemma 1.1. If L is uniformly parabolic in $\bar{Z}$ and the coefficients $b_p, C_{p,i}$ are continuous on Ψ_a and satisfies a uniform Hölder condition in $t\in\left[0,a\right]$. Then the Cauchy Problem (4.4) can be written in the abstract form (1.1) and (1.2), where the domain of the operator A is $S=B^{2k}\left(Z\right)\cap B_0^k\left(Z\right)$ and

$$Au=A\left(x,D\right)u=\sum_{|p|\leq 2k} a_p\left(x\right)D^p u, \qquad 4.8$$

where Ψ_a is the cylinder $\left\{\left(x,t\right):x\in Z,0<t<a\right\}$ for any $a\in\left[0,\infty\right]$ and by Υ_a the boundary $\left\{\left(x,t\right):x\in\partial Z,0<t<a\right\}$.

lemma 1.2. Let $E=L_2\left(Z\right)$ and assume that the functions $h_0\left(x\right),h_1\left(x\right),\ldots,h_{n-1}\left(x\right)$ be continuous, then the Cauchy Problem (4.4) exists and depends continuously on the initial conditions $h_0\left(x\right),h_1\left(x\right),\ldots,h_{n-1}\left(x\right)$.

PROOF. if $E=L_2\left(Z\right)$, then the domain $S=B^{2k}\left(Z\right)\cap B_0^k\left(Z\right)$ is dense in $L_2\left(Z\right)$. Consider the operator $H\left(t\right):=\sum_{|p|\leq 2k-1} b_p\left(x,t\right)D^p$, then the domain of the operators $H\left(t\right)$ and $H_i\left(t\right), i=1,2,3,\ldots\ldots,\nu$ can be chosen $B^{2k-1}\left(Z\right)\cap B_0^k\left(Z\right)$ which is dense in $L_2\left(Z\right)$.

Therefore, we can assume that

$$S_1=S_2=\ldots=S_\nu=B^{2k-1}\left(Z\right)\cap B_0^k\left(Z\right). \qquad 4.9$$

Since $Lu(x,t)$ is uniformly parabolic and the functions $h_0(x), h_1(x), \ldots, h_{n-1}(x)$ are continuous, then $A(x,D)$ can generate a semi-group $\{Q(t)\}$ of class C_0.

It is obvious that $\{Q(t)\}$ satisfies (1.7) and (1.8), and hence we can prove previous theorems and hence we can prove previous theorems can be applied for the Cauchy Problem (4.4).

From pervious lemma can conclude that the Cauchy Problem can be solved in

$$B^{2k-1}(Z) \cap B_0^{\ k}(Z), \qquad 4.10$$

without any restrictions on the characteristic forms of the operators $H(t)$ and $H_i(t)$ and depends only on $h_0(x), h_1(x), \ldots, h_{n-1}(x)$. (See [19-24]).

5. Conclusion

The Cauchy problem for some abstract differential equations is studied in the Banach space. The stability of solutions are proved under suitable conditions. An application for some partial differential equations is studied.

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